Math 5111 Measure Theory, Fall 2026

Basics

- Instructor: Luke Rogers
- Office hours: TTh TBA
- Classes: TTh 9:30-10:45 in MONT 227
- Course Description: General theory of measure and Lebesgue integration, L^p-spaces.
- Text: None required. Probably the easiest text to learn from is by Rich Bass and is available free online as a pdf: Bass Real Analysis for Graduate students
  Other standard texts at the level of this course include those of Folland (Real Analysis), Royden (Real Analysis), Rudin (Real and Complex Analysis). Each has strengths and weaknesses, and their material is not entirely overlapping.
- Assessment: 20% Homework, 30% Midterm, 50% Final , and up to 10% bonus for constructive input in class.

What is expected of you

My primary expectation is that you read and attempt to understand the material outside of class. This will involve reading and, especially, attempting to solve problems. My intent is that this will allow me to only lecture for part of the class time and for us to discuss problems in class that you have already thought about and worked on, so you are well-prepared to discuss them.

l suggest that this class should require a time commitment of at least 10 hours per week (including class time). In addition to discussing material in class, most of you will also find it useful to have some sort of study group with whom you will discuss concepts from the text and homework problems. Of course, the work you hand in must be your own.

I expect you to attend class, at least most of the time. I will not check whether you attend, nor will I have assessment items that require you to be in class, but there is honestly not much point in being in the course if you're not planning to show up. If you want to learn the material on your own, just read Bass' book and do all the exercises and you'll be fine. Class time

is where we'll try to make it easier and faster to get a fuller picture of the topic than you get from reading alone. In case it matters, I might call on you in class to explain what you understood the main points to be in the reading, or what you did on a problem, but your answer will not be assessed for course credit.

Assessment

I'm giving 20% for homework, 30% for the midterm and 50% for the final. This reflects the fact that the main goal of this course is to get you to the level in the topic that would allow you to pass the prelim. The midterm will be like half a prelim (half the material, half the number of questions, half the time) and the final will be at the level of a prelim. However, it won't be graded at the level of a prelim (in that you won't need a PhD pass level exam grade to get an A for the course).

The only way to get to the level where you'll pass a prelim is to do a bunch of problems at this level. This is where homework comes into the picture. Homework problems make up 20% of the course credit to give you incentive to do them each week, which is also why you have to hand them in on time. If you do them all correctly you should have no trouble getting most of the other 80% on the midterm and final exams. More to the point, if you do them incorrectly but learn how they should have been done you ought also to get most of the other 80%.

All your reading and homework assignments will be on HuskyCT. This also provides you with a due-date calendar.

Advice

I strongly encourage you to form study groups and/or discuss problems with your fellow students. One thing this does is normalizes the fact that sometimes the material is difficult for everyone. Another is that it helps you to be comfortable asking for more discussion of a topic in lecture. Finally, it helps you to solve problems.

I suggest that when you first look at a problem you try to solve it by yourself. Once you have your own solution, or have exhausted your efforts to find one, you are encouraged to discuss it with other students. Look at what at each person in the group has done, criticize it, try to find and fix the holes in each others proofs. And then remember that **the solution you hand in must be your own**!

Don't be afraid to try a lot of different approaches and ideas in solving a problem. It is not unusual to go through 20 pages or more of scratch work to

produce each page of homework solutions. Hands-on work with the material is where the learning really happens, so make it count.

Academic Misconduct, Plagiarism, etc

Quite simply, you should never hand in work that is not your own – whether you copied from a friend or the internet or a book or whatever. If you struggled with a problem, looked somewhere for an idea, and used that idea, then you should make a note on your solution of where you got help. If you have any doubt about the acceptable standards for academic conduct, consult the university's community standards:
UConn Academic Integrity page

At this level of course I don't expect to see any plagiarism, so I hope this section is redundant. In undergraduate courses I usually penalize a first offence with a 10% loss of course grade and a second offence with an additional 30% loss of course grade; more than that is an automatic F and a referral to academic standards. When homework is 20% of your grade, there should be no point in plagiarism – you'll lose more points on the exams for not really knowing the material than you'll gain on the homework.

Non-Academic Support

The university has many support services for the stresses and difficulties encountered by students. I urge you to make use of these early – don't wait and let a small problem get bigger before you deal with it!. Among the many offices some of you may find useful are the Dean of Students, Student Health and Wellness (the Mental Health division may be particulary helpful if you are stressed) and the Center for Students with Disabilities. It is worth looking at what each of these offices do, so you know where to find help if you or a friend needs it. I should also mention the Office of Institutional Equity, which deals with matters of discrimination and harassment; I hope you will not need their help, but it is good to be aware of the support they offer.

Course outline and learning goals – with rough schedule

There are several main parts to developing Lebesgue measure and integration, each of which includes multiple learning goals. The following list is long but perhaps not exhaustive. It has a tentative schedule, but don't be surprised if we need to diverge from it.

For each definition and theorem listed in the learning goals you should be able to precisely state the definition or theorem, including all hypotheses, give examples illustrating the definition or the use of the theorem, give examples illustrating why each hypothesis of a theorem is needed, and describe the intuitive content of the definition or theorem.

- Measures and σ-algebras (2 weeks) (Note: this doesn't take 2 weeks but I will give some preliminary review of Riemann integration and some background on why we are doing what we do in the course.)
  - Definition of σ-algebras, Borel σ-algebras
  - Definition of monotone class
  - Monotone class theorem
  - Existence of non-measurable sets in $\mathbb{R}$.
  - Definition of measure.
  - For μ a measure be able to prove: $E \subset F \implies \mu(E) \leq \mu(F)$ (monotonicity), $\mu(\cup E_j) \leq \sum_j \mu(E_j)$ (subadditivity), $E_j \subset E_{j+1}$ for all j implies $\mu(\cup E_j) = \lim_j \mu(E_j)$ (continuity from below), and $E_j \supset E_{j+1}$ for all j and $\mu(E_1) < \infty$ implies $\mu(\cap E_j) = \lim_j \mu(E_j)$ (continuity from above).
  - Definition of finite measure, a σ-finite measure, a null set.
  - Be able to construct the completion of a measure
  - Be able to construct the product σ-algebra on a product of measurable spaces.
- Integration (1 week)
  - Definition of a measurable function.
  - The composition in which a measurable function is followed by a continuous function results in a measurable function.
  - The collection of measurable functions is closed under elementary operations like addition, multiplication, max and min and taking the absolute value, as well as more complicated operations like

taking the sup, inf, lim sup or lim inf of a sequence of functions, as well as to the pointwise limit of a sequence of functions (provided this limit exists).

  - Definition of a simple function.
  - Be able to write a non-negative measurable function as a monotone increasing limit of simple functions.
  - Definition of the integral of a simple function
  - Integral of simple function is independent of how it is represented
  - Definition of the integral of a non-negative measurable function.
  - Be able to prove that a complex-valued measurable function can be decomposed into positive and negative real and imaginary parts that are measurable functions. Definition of $f \in L^1(\mu)$ and how the integral of an L^1 function is defined.
  - Be able to explain what is meant by equality a.e. of functions, verify that this is an equivalence relation and explain what is meant by considering L^1 as a quotient (vector) space by this relation. In particular, know that equality a.e. of L^1 functions occurs iff $\int |f - g| = 0$ and iff $\int_A f = \int_A g$ for all measurable A.

- Construction of measures (1 week)

  - Definition of an outer measure and how it differs from a measure.
  - Know the Caratheodory criterion for measurability and be able to state and use the corresponding theorem (that the measurable sets are a σ-algebra and the restriction of the outer measure to them is a measure). Understanding the connection between outer and inner approximation is useful here.
  - Caratheodory extension theorem.
  - Have a thorough understanding of the Caratheodory extension on the real line for the algebra of half-open intervals (half-open eliminates overlaps), for both Lebesgue and Riemann-Stieltjes measures.
  - Approximation properties of Lebegue measurable sets on $\mathbb{R}$ by open and closed sets. These are often called inner and outer regularity (this definition is used for Borel measures on any locally compact Hausdorff space).
  - Definition of G_δ and F_σ sets.

  - Be able to prove that any Lebesgue measurable set on $\mathbb{R}$ is contained in a G_δ set and contains an F_σ set, all three of which have the same measure.
  - Be able to prove that Lebesgue measure is invariant under rigid motion (i.e. maps that preserve the distance between any pair of points).

- Limit theorems (1 week)
  - Monotone convergence theorem (MCT)
  - Dominated convergence theorem (DCT)
  - Fatou's lemma.
  - Know how to derive each of the limit theorems from the others
  - Know some standard ways in which the limit theorems are used. One example: the fact that most basic properties of the integral are verified on simple functions and then extended to all integrable functions via limit theorems. (This approach is used for proving many other results too.)
  - Be able to transfer limit theorem results about sequences of functions to results about series of functions. A particularly nice example is: If $\sum_n \int |f_n| < \infty$ then $g = \sum_n |f_n|$ is in L^1 by monotone convergence, so it is finite a.e. and thus at a.e. x we have $\sum_n f_n(x)$ converges (absolutely), so $f = \sum_n f_n$ exists. Moreover, by dominated convergence with dominating function g one has $\int f = \sum_n \int f_n$.
  - $\int f(x,y)d\mu(x) \to \int f(x,y_0)d\mu(x)$ as $y \to y_0$ if there is an L^1 function g that dominates uniformly in y, i.e. $|f(x,y)| \leq g(x)$ for all x, y, and (similarly, by considering a difference quotient and using a mean value estimate) how to show that $\partial_y \int f(x,y)d\mu(x) = \int \partial_y f(x,y)d\mu(x)$ if one has an L^1 dominator h so $|\partial_y f(x,y)| \leq h(x)$ for all x, y.

- Modes of convergence (1 week)
  - Definitions of: uniform convergence, pointwise convergence, L^1 convergence, convergence a.e., convergence in measure
  - Uniform convergence implies pointwise implies a.e. convergence.

- Small (and trivial) but useful note: Anything about convergence of $f_n \to f$ may as well be about $f_n - f \to 0$, so can assume we're talking about $f_n \to 0$. Also, it is expected you know the undergrad analysis fact that that for each notion of convergence of a sequence there is a notion of Cauchy, and that convergent implies Cauchy. Sometimes we talk about, e.g., a sequence being Cauchy in measure or Cauchy in L^1.
- Be able to prove that Cauchy implies convergent for these notions of convergence. (i.e. completeness)
- Be able to prove that uniform and L^1 convergence both imply convergence in measure
- Be able to give examples showing none of the other convergences imply each other without further conditions. (Some standard examples are: moving mass off to infinity, concentrating mass at a point, moving mass back and forth along a line segment while concentrating it). You should also know what kinds of conditions are useful, some of which are limit theorems. E.g. cvge a.e and dominating function gives L^1 convergence (DCT).
- Know convergence in measure implies pointwise cvge a.e. for a subsequence. Thus L^1 convergence implies a.e. convergence of a subsequence as well.
- Know the Egoroff theorem: If $\mu(X) < \infty$ then convergence a.e. implies that for any $\epsilon > 0$ there is a set E_ϵ with $\mu(E) < \epsilon$ so one has uniform convergence on $X \setminus E_\epsilon$. For $f_m \to 0$ in measure this is done by showing $\cap_n \cup_{m \geq n} \{x : |f_m(x)| > \frac{1}{k}\}$ has measure 0 using continuity of the measure from above.

- Product measures (1 week)
  - If μ is a measure on $(X\mathcal{M})$ and ν is a measure on $(Y, \mathcal{N})$, know how to define $\mu \times \nu$ on the product σ-algebra $\mathcal{M} \times \mathcal{N}$ over $X \times Y$. Know this is unique if μ and ν are σ-finite and that bad things can happen if they are not.
  - Definition of the "slices" E_x and E^y for $E \in \mathcal{M} \times \mathcal{N}$ and $x \in X$, $y \in Y$. Also know when the functions $x \mapsto E_x$ and $y \mapsto E^y$ are measurable for their respective σ-algebras (this is a little tricky in general) and that $\nu \times \nu(E) = \int \mu(E^y) d\nu(y) = \int \nu(E_x) d\mu(x)$. You don't have to be able to give the proof, but (roughly) for

finite measures one uses the monotone class theorem and σ-finite ones are written as limits of finite ones via the MCT.

- Fubini-Tonelli theorem. This has two parts (Tonelli, for non-negative functions, and Fubini for L^1 functions) and justifies iterated integration. Each says that if f is in the relevant space on the product then $\int f(x,y)d\mu(x)$ and $\int f(x,y)d\nu(y)$ are each in the corresponding space as functions in the variable not yet integrated (a.e. in the Fubini case), and that $\int f d(\mu\times\nu) = \iint f(x,y)d\mu(x)d\nu(y) = \iint f(x,y)d\nu(y)d\mu(x)$ (product integration can be written as iterated integration). You don't have to know the proof, but this is done first on simple functions on products of intervals and then by approximation by simple functions and applying a limit theorem.
- A standard use of Fubini-Tonelli is as follows: One has f on $X\times Y$, measurable w.r.t. $\mathcal{M}\times\mathcal{N}$. Take $|f|$ and apply Tonelli: if using iterated integration in either order gives a finite number then you've established that $\int |f| d(\mu\times\nu) < \infty$ so f is in $L^1(\mu\times\nu)$ and the Fubini theorem is applicable to show $\int f d(\mu\times\nu)$ can be given by either iterated integral. In practice, what this means is that if you want to compute $\int f d(\mu\times\nu)$ then you first compute one or other of the iterated versions of the integral for $|f|$, then get to pick which one of the iterated integrals you want to use for f.
- Lebesgue measure on $\mathbb{R}^n$ as a product measure. It inherits inner and outer regularity from Lebesgue measure on $\mathbb{R}$. It can also be defined in other ways, like as n-dimensional Hausdorff measure. Some other properties, like how to prove the change of variables formula, may be discussed if time permits.

- Signed and complex measures; singularity, absolute continuity and the Radon-Nikodym theorem. (2 weeks)
  - Definition of signed measures and complex measures.
  - Hahn decomposition theorem. For this purpose one must know the definition of positive set and a negative set.
  - Jordan decomposition theorem.
  - Definition of two measures being mutually singular and being absolutely continuous.

- Radon-Nikodym theorem. This is a major theorem. It applies to a σ-finite measure μ and a measure ν absolutely continuous to μ. It says that $\nu = f\mu$ for a unique $f \in L^1(\mu)$ (recall that elements of L^1 are a.e. defined). We call f the Radon-Nikodym derivative of ν with respect to μ and write $f = \frac{d\nu}{d\mu}$.
- Lebesgue decomposition theorem. This says that for σ-finite measures μ, ν defined on the same σ-algebra one can uniquely decompose $\nu = \nu_a + \nu_s$, where ν_a is absolutely continuous to μ and ν_s is singular to μ.
- The preceding theorems are often combined as the Lebesgue-Radon-Nikodym theorem, telling you that for σ-finite μ, ν on the same σ-algebra one can write $\nu = f\mu + \nu_s$. It is very useful to get a concrete sense of this theorem by knowing examples that are Riemann-Stieltjes measures, like the sum of Lebesgue and Cantor measure on the unit interval. This also helps you understand why one should take the derivative of a measure with respect to a measure – you should be able to verify that the usual calculus derivative of a monotone function is the derivative of the a.c. part of the associated Riemann-Stieltjes measure with respect to the Lebesgue measure, for example.
- Definition of the total variation of a measure (via Jordan decomposition in the signed case and Radon-Nikodym or directly in the complex case).
- Lemma: $|\int f d\mu| \leq \int |f| d|\mu|$ and $|\mu + \nu| \leq |\mu| + |\nu|$, where $|\mu|$ is the total variation of μ and (here) μ and ν can be signed and/or complex valued.

- Differentiation on $\mathbb{R}$ (this is really what replaces the fundamental theorem of calculus (FTC)). (2 weeks)
  - Definition of the Hardy-Littlewood maximal function Mf of a locally integrable integrable function f (i.e. one for which $|f|$ has finite integral on compact sets)
  - Hardy-Littlewood maximal theorem: the measure of the set $\{x : Mf > \lambda\}$ is bounded by $\frac{C}{\lambda} \int |f|$.
  - Theorem: If f is integrable then $\int_a^x f dm$ is differentiable a.e. with derivative f (this is one part of FTC)
  - Definition of a function of bounded variation on $\mathbb{R}$

  - Be able to prove that a function of bounded variation is the difference of monotone functions
  - Be able to prove that a monotone function is differentiable a.e.
  - Definition of an absolutely continuous function on $\mathbb{R}$
  - Be able to prove that an absolutely continuous function is differentiable a.e. and is the integral of its derivative (this is the other part of FTC)
  - Know how to make examples (Cantor measure, Cantor function, Rieman-Stieltjes measures) that show that absolute continuity is needed
  - Be able to interpret the FTC results in terms of the Lebesgue-Radon-Nikodym theorem via Rieman-Stieltjes measures.

- L^p spaces (2 weeks)
  - Definition of a Banach space, a Hilbert space, and of L^p for $p \in (0, \infty]$
  - Be able to prove the Hölder, Minkowskii and Jensen inequalities
  - Be able to prove that L^p is a normed space for $p \geq 1$
  - Be able to prove that L^2 is an inner product space
  - Be able to prove completeness of L^p for $p \geq 1$
  - Be able to state and apply the Riesz representation theorem for L^p, $p \in [1, \infty)$

- Riesz representation theorem for bounded linear functionals on $C(X)$ (1 week)
  - Urysohn lemma
  - Riesz representation theorem that tells you bounded linear functionals on the continuous functions on either a compact or locally compact Hausdorff space X are given by integration against Radon measures (for which you should know the definition).